§9

9.1 (i) $|S_4| = 4! = 24 = 2^3.3$. So the Sylow 2-subgroups of S_4 are all the subgroups of S_4 of order 2^3. Let $P \in Syl_2 G$ where $G = S_4$. Then $n_2 = [G : N_G(P)]$ divides 3 (as $P \leqslant N_G(P) \leqslant G$).

From q 7.6 (iii) $\langle(1234),(13)\rangle$, $\langle(1324),(34)\rangle$ and $\langle(1243),(23)\rangle$ are three subgroups of G of order $2^3 = 8$.

$$\therefore\ Syl_2 G = \{\langle(1234),(13)\rangle, \langle(1324),(34)\rangle, \langle(1243),(23)\rangle\}.$$

The Sylow 3-subgroups of S_4 are all the subgroups of S_4 of order 3. $\therefore$ (for $G = S_4$)

$$Syl_3 G = \{\langle(123)\rangle, \langle(124)\rangle, \langle(134)\rangle, \langle(234)\rangle\}$$

(ii) Let $G = A_5$. Then $|G| = \frac{5!}{2} = 60 = 2^2.3.5$. Since $\langle(12)(34), (13)(24)\rangle = \{(1), (12)(34), (13)(24), (14)(23)\}$ is a subgroup of G

of order 4,

$$P = \langle (12)(34), (13)(24) \rangle \in Syl_2\, G.$$

Let H be the stabilizer in G of 5 $(\in \Omega = \{1,2,3,4,5\})$. Then $H \leqslant G$ (Lemma 5.5). Also $P \leqslant H$ and $H \cong A_4$. So $P \trianglelefteq H$ (by structure of A_4). $\therefore$ $H \leqslant N_G(P)$ and so $|N_G(P)| = 12$ or 60. By calculation (or using the fact that A_5 is simple) $|N_G(P)| \neq 60$. $\therefore$ $|N_G(P)| = 12$ and so $n_2 = [G : N_G(P)] = \dfrac{|G|}{|N_G(P)|} = 5$. Hence

$$Syl_2\, G = \{ \langle (12)(34), (13)(24) \rangle, \langle (12)(35), (13)(25) \rangle, \langle (12)(45), (14)(25) \rangle, \langle (13)(45), (14)(35) \rangle, \langle (23)(45), (24)(35) \rangle \}.$$

9.2 (i) First show that P is a subgroup of $G = GL(2,p)$.

Since $\begin{pmatrix}1&0\\0&1\end{pmatrix} \in P$, $P \neq \phi$. Since $\det\begin{pmatrix}1&a\\0&1\end{pmatrix} = 1$, $\begin{pmatrix}1&a\\0&1\end{pmatrix} \in G$. So $\phi \neq P \subseteq G$.

Let $\begin{pmatrix}1&a\\0&1\end{pmatrix}, \begin{pmatrix}1&b\\0&1\end{pmatrix} \in P$ (so $a, b \in F$).

Since $\begin{pmatrix}1&b\\0&1\end{pmatrix}^{-1} = \begin{pmatrix}1&-b\\0&1\end{pmatrix}$,

$$\begin{pmatrix}1&a\\0&1\end{pmatrix}\begin{pmatrix}1&b\\0&1\end{pmatrix}^{-1} = \begin{pmatrix}1&a\\0&1\end{pmatrix}\begin{pmatrix}1&-b\\0&1\end{pmatrix} = \begin{pmatrix}1&-b+a\\0&1\end{pmatrix} \in P$$

as $a, b \in F$ implies $-b+a \in F$.

Therefore $P \leq G$ by the subgroup criterion.

Also $|P| = |F| = p$.

By q 2.5 $|GL(2,p)| = (p^2-1)(p^2-p)$
$$= p(p-1)(p^2-1)$$

Note that $p \nmid (p-1)$ and $p \nmid (p^2-1)$, and so $p \nmid (p-1)(p^2-1)$. $\therefore$ $|G| = pm$ with $p \nmid m$ ($m = (p-1)(p^2-1)$). Hence $P \in Syl_p G$.

(ii) Suppose $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in N_G(P)$ $(\alpha, \beta, \gamma, \delta \in F)$

Let $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \in P$. Set $d = \det\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} =$ $\alpha\delta - \beta\gamma$. Recall $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^{-1} = \frac{1}{d}\begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}$.

Then $g^{-1}xg = \frac{1}{d}\begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$

$$= \frac{1}{d}\begin{pmatrix} \delta\alpha + \delta a\gamma - \beta\gamma & \delta^2 a \\ -\gamma^2 a & -\gamma\beta - \gamma a\delta + \alpha\delta \end{pmatrix}$$

which must be in P $\forall a \in F$. So $-\gamma^2 a = 0$ $\forall a \in F$ which implies $\gamma = 0$. So $d = \alpha\delta$

$$\therefore g^{-1}xg = \frac{1}{d}\begin{pmatrix} \delta\alpha & \delta^2 a \\ 0 & \alpha\delta \end{pmatrix} = \begin{pmatrix} 1 & \delta^2 a/d \\ 0 & 1 \end{pmatrix} \quad (*)$$

So $N_G(P) \subseteq \left\{ \begin{pmatrix} b & a \\ 0 & c \end{pmatrix} \middle| \, a, b, c \in F,\ b \neq 0 \neq c \right\}$

and by $(*)$ this containment is an equality.

(iii) By (ii)

$$|N_G(P)| = p(p-1)(p-1).$$

So $n_p = [G : N_G(P)] = \dfrac{|G|}{|N_G(P)|} =$

$$\frac{p(p-1)(p^2-1)}{p(p-1)(p-1)} = p+1.$$

9.3 Suppose $R \trianglelefteq G$, R a p-subgroup of G. Let $P \in \mathrm{Syl}_p G$. By Thm 9.1(iii) $R \leq P_1$ for some $P_1 \in \mathrm{Syl}_p G$. By Theorem 9.1(ii) $\exists$ $g \in G$ such that $P_1^g = P$.

$\therefore \quad R \underset{\substack{\uparrow \\ R \trianglelefteq G}}{=} R^g \leq P_1^g = P$, and so R is contained in every Sylow p-subgroup of G.

9.4 (i) Since $P \cap N$ is a p-subgroup of N, $P \cap N \leq R$ where $R \in \mathrm{Syl}_p N$ by Theorem 9.1 (iii) (applied to N). Likewise, as R is a p-subgroup of G, $R \leq P_1$ for some $P_1 \in \mathrm{Syl}_p G$.

Now $R \leq N \cap P_1 \leq N$. So, since $R \in \mathrm{Syl}_p N$

and $N \cap P_1$ is a p-subgroup of N, $R = N \cap P_1$.

By Theorem 9.1(ii) $\exists\, g \in G$ such that $P_1^g = P$.

$\therefore\ R^g = (N \cap P_1)^g = N^g \cap P_1^g \underset{N \trianglelefteq G}{=} N \cap P.$

Hence $|R| = |R^g| = |N \cap P|$. Thus, as $P \cap N \leq R$, $R = P \cap N$. $\therefore\ P \cap N \in Syl_p N$.

(ii) Let $|G| = p^a m$ and $|N| = p^b n$ where $p \nmid m$ and $p \nmid n$. Since $P \cap N \in Syl_p N$ by (i), $|P \cap N| = p^b$.

Also $|G/N| = p^{a-b}(m/n)$ and $p \nmid (m/n)$.

From $PN/N \cong P/P \cap N$ (Theorem 7.17), we get PN/N is a p-subgroup of G/N of order $|P|/|P \cap N| = p^a/p^b = p^{a-b}$.

$\therefore\ PN/N \in Syl_p\, G/N.$

(iii) $G = S_3$, $H = \langle (12) \rangle$, $P = \langle (23) \rangle$,

Then $P \in Syl_2 G$ and $P \cap H = \{(1)\} \notin Syl_2 H$.

9.5 Suppose $|G| = p^2 q$ and G is simple.

By Theorem 9.1 (iv) $n_p \mid q$. $\therefore$ $n_p = 1$ or q.

G simple $\Rightarrow n_p \neq 1$ and so $n_p = q$.

$(*)$ Let $P \in \mathrm{Syl}_p G$ and let $1 \neq x \in P$. Then $C_G(x) = P$.

Since $|P| = p^2$, P is abelian by Lemma 7.7 and so $P \leq C_G(x)$. Lagrange's theorem $\Rightarrow$ either $P = C_G(x)$ or $G = C_G(x)$. If $G = C_G(x)$, then $1 \neq \langle x \rangle \trianglelefteq G$ with $\langle x \rangle \leq P \neq G$, contrary to G being simple. $\therefore$ $(*)$ holds.

$(**)$ Let $P_1, P_2 \in \mathrm{Syl}_p G$ with $P_1 \neq P_2$. Then $P_1 \cap P_2 = 1$.

If $P_1 \cap P_2 \neq 1$, then $\exists\, 1 \neq x \in P_1 \cap P_2$. By $(*)$ twice

$$P_1 = C_G(x) = P_2, \text{ whereas } P_1 \neq P_2.$$

Thus $P_1 \cap P_2 = 1$.

Also we have

(***) For $Q_1, Q_2 \in Syl_q G$ with $Q_1 \neq Q_2$, $Q_1 \cap Q_2 = 1$.

(***) holds because $|Q_1| = q = |Q_2|$.

Using (**) and (***), by counting elements of G in Sylow p- and Sylow q-subgroups we get

$$|G| = p^2 q \geq n_p(p^2-1) + n_q(q-1) + 1$$
$$= q(p^2-1) + n_q(q-1) + 1$$
$$\Rightarrow \quad q-1 \geq n_q(q-1)$$

$\therefore n_q = 1$, contrary to G being simple.

$\therefore$ G is not simple.

9.6 (i) Suppose $|G| = 56 = 2^3.7$ and G is simple.
So $n_7 \neq 1$ and hence, by Theorem 9.1 (iv), $n_7 = 8$

Set $L = \bigcup_{P \in Syl_7 G} P \setminus \{1\}$. Then $|L| = 6n_7 = 48$

(because for $P_1, P_2 \in Syl_7 G$, $P_1 \neq P_2 \Rightarrow P_1 \cap P_2 = 1$)

Let $Q \in Syl_2 G$. So $|Q| = 2^3$ and $Q \cap L = \phi$.
Hence $Q = G \setminus L$ and so $n_2 = 1$, contrary to G being simple.

(ii) Suppose $|G| = 3^2.5.7$ and G is simple.
So $n_3 \neq 1$. Since $n_3 \equiv 1 \pmod 3$ and $n_3 \mid 5.7$, we must have $n_3 = 7$. Let $Q \in Syl_3 G$. Then $|N_G(Q)| = 3^2.5$. Now let $P \in Syl_5 N_G(Q)$.
So $|P| = 5$ and hence $P \in Syl_5 G$. Applying Sylow's theorem to $N_G(Q)$ for the prime 5 we get $|Syl_5 N_G(Q)| = 1$. Thus $P \trianglelefteq N_G(Q)$ and so $N_G(Q) \leq N_G(P)$. $\therefore n_5 = [G : N_G(P)] = 1$ or 7.
Since $7 \not\equiv 1 \pmod 5$, we have $n_5 = 1$, contrary to G being simple.

(iii) Suppose $|G| = 2^3.3.7.23$ and G is simple. Since $n_{23} \mid 2^3.3.7$, $n_{23} \neq 1$ and

$n_{23} \equiv 1 \pmod{23}$, we deduce $n_{23} = 24 = 2^3.3$.

Let $Q \in Syl_{23}\, G$. Then $|N_G(Q)| = 7.23$. Let $P \in Syl_7\, N_G(Q)$. Then just as in (ii) get $N_G(Q) \leq N_G(P)$. $\therefore$ $n_7 = [G : N_G(P)]$ divides $2^3.3$. Since $n_7 \neq 1$ and $n_7 \equiv 1 \pmod 7$, this implies $n_7 = 8$. Hence $[G : N_G(P)] = 8$.

By q 8.2 (ii) $|G|$ divides $8!$. But $23 \mid |G|$ and $23 \nmid 8!$, a contradiction.