§8

8.1 Since G is abelian, all subgroups of G are normal. Therefore, as G is simple, the only subgroups of G are $\{1\}$ and G. So for $1 \neq g \in G$, as $\langle g \rangle \leqslant G$, we have $\langle g \rangle = G$. Hence g has prime order, say q, and then $G \cong \mathbb{Z}_q$

8.2 (i) From lectures: $\exists$ homomorphism $\theta : G \longrightarrow S_\Omega$ where $\Omega = \{Hx \mid x \in G\}$ and $\ker\theta = \bigcap_{x \in G} H^x$. So $\ker\theta \leqslant H \neq G$. Since G is simple and $\ker\theta \trianglelefteq G$, we must have $\ker\theta = 1$.

$$\therefore \; G \cong G/\{1\} = G/\ker\theta \underset{\substack{\uparrow \\ \text{FIRST ISOMORPHISM} \\ \text{THEOREM}}}{\cong} \operatorname{im}\theta \leqslant S_\Omega$$

$\therefore$ G is isomorphic to a subgroup of $S_\Omega = S_n$ (namely $\operatorname{im}\theta$)

Solution 8.3(i)

$i = 1, 2, 3, 4$ should be

$i = 0, 1, 2, 3$

(ii) Since $|S_n| = n!$, Lagrange's theorem and part (i) imply $|G| \mid n!$.

8.3 (i) $G = \mathrm{Dih}(8) = \{a^i b^j \mid i = 1,2,3,4;\ j = 0,1\}$

$a = \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}$ $b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\lambda = e^{2\pi i/4}$

Take $G_1 = \langle a \rangle\ (\cong \mathbb{Z}_4)$, $G_2 = \langle a^2 \rangle\ (\cong \mathbb{Z}_2)$, $G_1 = 1$.

Since $[G : G_1] = 2$, $G_1 \trianglelefteq G$; $a^2 \in Z(G)$ and so $G_2 \trianglelefteq G_1$. Composition factors: $\mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2$

(ii) $G_1 = A_4$, $G_2 = \langle (12)(34), (13)(24) \rangle$, $G_3 = \langle (12)(34) \rangle$, $G_4 = 1$. $[G : G_1] = 2 \Rightarrow G_1 \trianglelefteq G$, and $G_2 \trianglelefteq G \Rightarrow G_2 \trianglelefteq G_1$. Since G_2 is abelian, $G_3 \trianglelefteq G_2$.

Composition factors: $G/G_1 \cong \mathbb{Z}_2$, $G_1/G_2 \cong \mathbb{Z}_3$, $G_2/G_3 \cong \mathbb{Z}_2 \cong G_3/G_4$.

(iii) $G_1 = A_5$, $G_2 = 1$. Since $[G : G_1] = 2$, $G_1 \trianglelefteq G$. Also A_5 is simple (Lemma 8.4)

Composition factors: $G/G_1 \cong \mathbb{Z}_2$, $G_1/G_2 \cong A_5$